Answering a problem with plain English

Problem 1.8c: For a solid, we also define the **linear thermal expansion coefficient**, α, as the fractional increase in length per degree:

$$\alpha \equiv \frac{\Delta L/L}{\Delta T}$$

(c) Prove that the volume thermal expansion coefficient of a solid is equal to the sum of its linear expansion coefficients in the three directions: $\beta = \alpha_x + \alpha_y + \alpha_z$. (So for an isotropic solid, which expands the same in all directions, $\beta = 3\alpha$.)

Traditional solution

$\beta \equiv \frac{\Delta V/V}{\Delta T}$, and $\Delta V = V_f - V_i$
$V_i = xyz$
Increase temperature by $\Delta T \Rightarrow V_f = (x + \Delta x)(y + \Delta y)(z + \Delta z)$
To leading order in Δx, etc $V_f \simeq xyz + \Delta xyz + x\Delta yz + xy\Delta z$
$\Delta V/V = \Delta x/x + \Delta y/y + \Delta z/z$
$\beta \equiv \frac{\Delta V/V}{\Delta T} \simeq \frac{\Delta x/x+\Delta y/y+\Delta z/z}{\Delta T} = \alpha_x + \alpha_y + \alpha_z$ ■

Plain English solution

We want to understand how the volume thermal expansion coefficient should be related to the linear thermal expansion coefficient. These expansion coefficients are so far empirical properties of a material (we haven't related them to changes in the interatomic interactions, for example) but we can safely assume that the fractional expansion will be small. It is simple to write down the total change in volume of an object of side lengths x, y, z subject to a change in temperature of ΔT as

$$\Delta V \equiv V_f - V_i = (x + \Delta x)(y + \Delta y)(z + \Delta z) - xyz.$$

However since the change in any of the linear dimensions of an object subject to a temperature change will be small we can safely ignore any term that has more than one Δ in it, as it will contribute negligibly to the whole. Therefore, we can approximate the total change in volume by the sum of changes in volume for only one dimension at a time as

$$\Delta V \simeq \Delta xyz + x\Delta yz + xy\Delta z$$

and thus

$$\Delta V/V \simeq \Delta x/x + \Delta y/y + \Delta z/z.$$

From here the answer is easy. We are interested in the volume thermal expansion coefficient β, so plugging in our above approximation we find that

$$\beta \equiv \frac{\Delta V/V}{\Delta T} \simeq \frac{\Delta x/x + \Delta y/y + \Delta z/z}{\Delta T} \equiv \alpha_x + \alpha_y + \alpha_z$$

as expected ■.